Online Appendix to "Trading Networks with Frictions"

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D Proof of Theorem A.1

It is clear that SFS $\implies$ FS. Hence, it suffices to prove that FS $\implies$ WQ and that WQ $\implies$ SFS. We first extend the demand correspondence to allow for some infinite prices (as in the definition of WQ in Appendix A) and formulate full substitutability in terms of the (extended) demand correspondence. We then derive relationships between the (extended) demand correspondence and the indirect utility function. Finally, we use these relationships to prove the two desired implications.

For the remainder of the proof, we fix a firm $f \in F$ whose preferences we analyze.

D.1 The extended demand correspondence

Given a price vector $p \in \mathsf{P}^f$ and a set $\Xi \subseteq \Omega_f$ of trades, let

$$U^f(\Xi|p) = u^f\left(\Xi, \left(p_{\Xi_{f\to}}, (-p)_{\Xi_{\to f}}, 0_{\Omega_f \smallsetminus \Xi}\right)\right)$$

denote f's utility of making the trades in Ξ of trades at prices given by p, where we write $u^f(\Xi, t) = -\infty$ if $t_\omega = -\infty$ for any $\omega \in \Omega_f$. Define the *extended demand correspondence* $\mathbf{D}^f : \mathsf{P}^f \rightrightarrows \mathcal{P}(\Omega_f)$ by

$$\mathbf{D}^f(p) = \operatorname*{arg\,max}_{\Xi \subseteq \Omega_f} U^f(\Xi|p).$$

Note that the restriction of the extended demand correspondence to $\mathbb{R}^{\Omega_f}$ is the demand correspondence D^f. Berge's Maximum Theorem guarantees that $\mathbf{D}^f$ is upper hemi-continuous. We can apply a perturbation argument to show that $\mathbf{D}^f$ is generically single-valued.

Claim D.1. *The set* $\{p \in \mathsf{P}^f \mid |\mathbf{D}^f(p)| = 1\}$ *is open and dense in* P^f.

Proof. Let $\mathfrak{S}_f = \{p \in \mathsf{P}^f \mid |\mathbf{D}^f(p)| = 1\}$. The set $\mathfrak{S}_f$ is open because $\mathbf{D}^f$ is upper hemi-continuous and non-empty-valued and $\mathcal{P}(\Omega_f)$ is discrete. To see that $\mathfrak{S}_f$ is dense, note that for all $\Xi \neq \Xi' \subseteq \Omega_f$, the set

$$\{p \in \mathsf{P}^f \mid U^f(\Xi|p) = U^f(\Xi'|p) \neq -\infty\}$$

is nowhere dense. Indeed, if $U^f(\Xi|p) = U^f(\Xi'|p) \neq -\infty$, we have that $U^f(\Xi|p') \neq U^f(\Xi'|p')$ for all $p' \in \mathsf{P}^f$ of the form $p' = \left(p_{\Omega \smallsetminus \{\omega\}}, p_\omega + \epsilon\right)$ with $\epsilon > 0$ and $\omega \in$

$(\Xi \smallsetminus \Xi') \cup (\Xi' \smallsetminus \Xi)$. $\square$

In the course of the proof of Theorem A.1, it will be useful to express full substitutability as a condition on the (extended) demand correspondence. We therefore write full substitutability in demand language similarly to Hatfield et al. (2019).

Assumption D.1 (Full substitutability in demand language—FS-D)**.** For all $p \leq p' \in \mathsf{P}^f$, if $\mathbf{D}^f(p) = \{\Xi\}$ and $\mathbf{D}^f(p') = \{\Xi'\}$, then we have that

$$\begin{aligned} \Xi' \cap \{\omega \in \Omega_{f\to} \mid p_\omega = p'_\omega\} \subseteq \Xi \\ \Xi \cap \{\omega \in \Omega_{\to f} \mid p_\omega = p'_\omega\} \subseteq \Xi' \end{aligned}.$$

We also write the constitutent conditions of strong full substitutability in demand language similarly to Hatfield et al. (2019).

Assumption D.2 (Increasing-price full substitutability for sales in demand language—IFSS-D)**.** For all $p \leq p' \in \mathsf{P}^f$ and $\Xi \in \mathbf{D}^f(p)$, there exists $\Xi' \in \mathbf{D}^f(p')$ with

$$\Xi' \cap \{\omega \in \Omega_{f\to} \mid p_\omega = p'_\omega\} \subseteq \Xi.$$

Assumption D.3 (Decreasing-price full substitutability for sales in demand language—DFSS-D)**.** For all $p \geq p' \in \mathsf{P}^f$ and $\psi \in \Xi \in \mathbf{D}^f(p)$ with $\psi \in \Omega_{f\to}$ and $p_\psi = p'_\psi$, there exists $\Xi' \in \mathbf{D}^f(p')$ with $\psi \in \Xi'$.

The substitutability conditions in choice language are equivalent to their demand-language analogues, as the following result shows formally.

Claim D.2. *FS (resp. IFSS, DFSS) is equivalent to FS-D (resp. IFSS-D, DFSS-D).*

Proof. Given a finite set of contracts $Y \subseteq X$, define a price vector $p_f(Y) \in \mathbb{R}^{\Omega_f}$ by

$$p_f(Y)_\omega = \begin{cases} \sup\limits_{(\omega,q)\in Y} q & \text{for } \omega \in \Omega_{f\to} \\ \inf\limits_{(\omega,q)\in Y} q & \text{for } \omega \in \Omega_{\to f} \end{cases},$$

so $p_f(Y)_\omega$ is the most favorable price at which ω is available in Y. It follows from the definitions of C^f and $\mathbf{D}^f$ that

$$C^f(Y) = \big\{\{(\omega, p_f(Y)_\omega) \mid \omega \in \Psi\} \mid \Psi \in \mathbf{D}^f\left(p_f(Y)\right)\big\}$$

for all finite sets $Y \subseteq X$. It follows that FS-D (resp. IFSS-D, DFSS-D) implies FS (resp. IFSS, DFSS), as claimed. The definitions of C^f and $\mathbf{D}^f$ also imply that

$$\mathbf{D}^f(p) = \left\{\tau(Y) \mid Y \in C^f\left(\{(\omega, p_\omega) \mid p_\omega \in \mathbb{R}\}\right)\right\}$$

for all $p \in \mathsf{P}^f$. It follows that FS (resp. IFSS, DFSS) implies FS-D (resp. IFSS-D, DFSS-D), as claimed. □

D.2 Relating the indirect utility function to demand

We next relate the indirect utility function to properties of the (extended) demand correspondence. We use these relationships in the proof of Theorem A.1.

Throughout this section, we use a monotonicity property of the indirect utility function. Specifically, the monotonicity of the utility function in transfers implies that

$$V^f(p) \geq V^f(q) \text{ whenever } p_{\Omega_{f\to}} \geq q_{\Omega_{f\to}} \text{ and } p_{\Omega_{\to f}} \leq q_{\Omega_{\to f}}. \tag{D.1}$$

Our first claim shows that there is a selection from the (extended) demand correspondence in which no sale in a set Γ of sales is demanded if and only if some (or every) reduction in the prices of all sales in Γ does not reduce f's indirect utility.

Claim D.3. *Let $p \in \mathsf{P}^f$ and let $\Gamma \subseteq \Omega_{f\to}$ be such that $p_\omega \in \mathbb{R}$ for all $\omega \in \Gamma$. The following are equivalent.*

(1) There exists $\Xi \in \mathbf{D}^f(p)$ with $\Gamma \cap \Xi = \varnothing$.

(2) For all $r \in \mathsf{P}^f$ with $r_{\Omega_f \smallsetminus \Gamma} = p_{\Omega_f \smallsetminus \Gamma}$, we have that $V^f(r) \geq V^f(p)$.

(3) There exists $r \in \mathsf{P}^f$ such that $r_{\Omega_f \smallsetminus \Gamma} = p_{\Omega_f \smallsetminus \Gamma}$, $r_\omega < p_\omega$ for all $\omega \in \Gamma$, and $V^f(r) \geq V^f(p)$.

Proof. We first show that (1) $\implies$ (2). Suppose that $\Xi \in \mathbf{D}^f(p)$ is such that $\Gamma \cap \Xi = \varnothing$. Let $r \in \mathsf{P}^f$ such that $r_{\Omega_f \smallsetminus \Gamma} = p_{\Omega_f \smallsetminus \Gamma}$. Note that $p_\Xi = r_\Xi$, and hence we have that $U^f(\Xi|p) = U^f(\Xi|r)$. By the definition of V^f, it follows that

$$V^f(p) = U^f(\Xi|p) = U^f(\Xi|r) \leq V^f(r),$$

as desired.

It is clear that (2) $\Longrightarrow$ (3). We next show that (3) $\Longrightarrow$ (1). Suppose that $r \in \mathsf{P}^f$ is such that $r_{\Omega_f \smallsetminus \Gamma} = p_{\Omega_f \smallsetminus \Gamma}$, $r_\omega < p_\omega$ for all $\omega \in \Gamma$, and $V^f(r) \geq V^f(p)$. Let $\Xi \in \mathbf{D}^f(r)$ be arbitrary. Due the monotonicity of the utility function in transfers, we must have that $U^f(\Xi|r) \leq U^f(\Xi|p)$ with equality if and only if $\Gamma \cap \Xi = \varnothing$. By the definition of V^f, it follows that

$$U^f(\Xi|p) \leq V^f(p) \leq V^f(r) = U^f(\Xi|r) \leq U^f(\Xi|p).$$

Therefore, we must have that $V^f(p) = U^f(\Xi|p) \leq U^f(\Xi|r)$, and hence that $\Xi \in \mathbf{D}^f(p)$ and that $\Gamma \cap \Xi = \varnothing$. In particular, there exists $\Xi \in \mathbf{D}^f(p)$ with $\Gamma \cap \Xi = \varnothing$, as desired. □

Our second claim shows that a sale ω is demanded in some selection from the (extended) demand correspondence if and only if every increase in the price of ω raises f's indirect utility.

Claim D.4. *Let $p \in \mathsf{P}^f$ and let $\omega \in \Omega_{f\to}$ be such that $p_\omega \in \mathbb{R}$. The following are equivalent.*

(1) There exists $\Xi \in \mathbf{D}^f(p)$ with $\omega \in \Xi$.

(2) We have that $V^f(r) > V^f(p)$ for all $p < r \in \mathsf{P}^f$ with $r_{\Omega_f \smallsetminus \{\omega\}} = p_{\Omega_f \smallsetminus \{\omega\}}$.

Proof. We first show that (1) $\Longrightarrow$ (2). Let $\Xi \in D^f(p)$ be such that $\omega \in \Xi$. Suppose that $p < r \in \mathsf{P}^f$ is such that $r_{\Omega_f \smallsetminus \{\omega\}} = p_{\Omega_f \smallsetminus \{\omega\}}$. Due to the monotonicity of the utility function in transfers, we must have that $U^f(\Xi|r) > U^f(\Xi|p)$. By the definition of V^f, we have that

$$V^f(p) = U^f(\Xi|p) < U^f(\Xi|r) \leq V^f(r).$$

We next show that (2) $\Longrightarrow$ (1). We actually prove the contrapositive. Suppose that $\omega \notin \Xi$ for all $\Xi \in \mathbf{D}^f(p)$. Due to the upper hemi-continuity of $\mathbf{D}^f$ and the discreteness of $\mathcal{P}(\Omega_f)$, there is an open neighborhood $\mathfrak{V} \subseteq \mathsf{P}^f$ of p such that $\mathbf{D}^f(r) \subseteq \mathbf{D}^f(p)$ for all $r \in \mathfrak{V}$. Let $\epsilon > 0$ be such that $r = \left(p_{\Omega_f \smallsetminus \{\omega\}}, (p+\epsilon)_\omega\right) \in \mathfrak{V}$, and let $\Xi \in \mathbf{D}^f(r) \subseteq \mathbf{D}^f(p)$. As $\omega \notin \Xi$, we have that $U^f(\Xi|r) = U^f(\Xi|p)$. By the definition of V^f, it follows that

$$V^f(p) = U^f(\Xi|p) = U^f(\Xi|r) = V^f(r).$$

In particular, we have that $V^f(r) \leq V^f(p)$. □

D.3 Proof that FS $\Longrightarrow$ WQ

We first prove that $V^f(p \vee q) > V^f(q) \implies V^f(p) > V^f(p \wedge q)$ whenever $p, q \in \mathsf{P}^f$ are such that $p_{\Omega_{f\to}} \leq q_{\Omega_{f\to}}$ or $p_{\Omega_{\to f}} \leq q_{\Omega_{\to f}}$. We prove the contrapositive. Suppose that $V^f(p) \leq V^f(p \wedge q)$; we prove that $V^f(p \vee q) \leq V^f(q)$.

If $p_{\Omega_{f\to}} \leq q_{\Omega_{f\to}}$, then we have that $(p \vee q)_{\Omega_{f\to}} = q_{\Omega_{f\to}}$. As $p \vee q \geq q$, (D.1) implies that $V^f(p \vee q) \leq V^f(q)$, as desired. Hence, we can assume that $p_{\Omega_{\to f}} \leq q_{\Omega_{\to f}}$.

Define a set of trades by

$$\Gamma = \{\omega \mid p_\omega > q_\omega\} \subseteq \Omega_{f\to}.$$

Note that

$$\Gamma = \{\omega \mid p_\omega > (p \wedge q)_\omega\} = \{\omega \mid (p \vee q)_\omega > q_\omega\}.$$

Since $V^f(p \wedge q) \geq V^f(p)$, the (3) $\Longrightarrow$ (1) implication of Claim D.3 guarantees that there exists $\Xi \in \mathbf{D}^f(p)$ such that $\Gamma \cap \Xi = \varnothing$.

To complete the argument, we perturb p and q to move to the locus on which the extended demand correspondence is single-valued, and then use Claim D.3 and full substitutability in demand language to conclude that $V^f(q) \geq V^f(p \vee q)$. More formally, because $\mathbf{D}^f$ is upper hemi-continuous and $\mathcal{P}(\Omega_f)$ is discrete, there exists $\epsilon > 0$ be such that $\mathbf{D}^f(p+s) \subseteq \mathbf{D}^f(p)$ and $\mathbf{D}^f(p \vee q + s) \subseteq \mathbf{D}^f(p \vee q)$ for all $s \in \mathbb{R}^{\Omega_f}$ with $\|s\| < 2|\Omega_f|\epsilon$. Consider the price change vector

$$s = \left((-\epsilon)_{\Omega_{f\to} \smallsetminus \Xi}, 0_{\Omega_{\to f} \cup \Xi}\right).$$

The monotonicity of u^f in transfers implies that

$$U^f\left(\Psi | p\right) \geq U^f\left(\Psi | p+s\right)$$

for all $\Psi \subseteq \Omega_f$, with equality if and only if $\Psi_{f\to} \subseteq \Xi$. It follows that $\Psi_{f\to} \subseteq \Xi \subseteq \Omega_f \smallsetminus \Gamma$ for all $\Psi \in \mathbf{D}^f(p+s)$. Due to Claim D.1 and the upper hemi-continuity of $\mathbf{D}^f$, there exists $s' \in \mathbb{R}^{\Omega_f}$ with $\|s'\| < \epsilon$ such that $|\mathbf{D}^f(p+s+s')| = |\mathbf{D}^f(p \vee q + s + s')| = 1$ and $\mathbf{D}^f(p \vee q + s + s') \subseteq \mathbf{D}^f(p \vee q)$. Let $\mathbf{D}^f(p+s+s') = \{\Psi\}$. By construction, we have that $\|s+s'\| < 2|\Omega_f|\epsilon$ and that $\Psi \cap \Gamma = \varnothing$.

Let $\mathbf{D}^f(p \vee q + s + s') = \{\Psi'\}$. Note that $p_\Gamma = (p \vee q)_\Gamma$ by construction. Claim D.2 implies that FS-D must be satisfied, and FS-D guarantees that $\Psi' \cap \Gamma \subseteq \Psi$. Hence, we have that $\Psi' \cap \Gamma \subseteq \Psi \cap \Gamma = \varnothing$. By construction, we have that $\Psi' \in \mathbf{D}^f(p \vee q + s) \subseteq \mathbf{D}^f(p)$. By the (1) $\implies$ (2) implication of Claim D.3, it follows that $V^f(q) \geq V^f(p \vee q)$.

Analogous logic, exchanging the roles of purchases and sales, shows that $V^f(p \wedge q) > V^f(p) \implies V^f(q) > V^f(p \vee q)$ whenever $p, q \in \mathsf{P}^f$ are such that $p_{\Omega_{f\to}} \leq q_{\Omega_{f\to}}$ or $p_{\Omega_{\to f}} \leq q_{\Omega_{\to f}}$.

D.4 Proof that WQ $\implies$ SFS

By symmetry, it suffices to prove that WQ implies IFSS and DFSS. In light of Claim D.2, it therefore suffices to prove that WQ implies IFSS-D and DFSS-D.

We first prove that WQ implies IFSS-D. To prove this implication, we exploit Claim D.3. Formally, let $p \leq p' \in \mathsf{P}^f$ and let $\Xi \in \mathbf{D}^f(p)$. Define a set of trades by

$$\Gamma = \{\omega \in \Omega_{f\to} \smallsetminus \Xi \mid p_\omega = p'_\omega \in \mathbb{R}\}.$$

Define a price change vector $s = ((-1)_\Gamma, 0_{\Omega_f \smallsetminus \Gamma})$. By construction, we have that $\Gamma \cap \Xi = \varnothing$ and that $p_\omega \in \mathbb{R}$ for all $\omega \in \Gamma$. Therefore, the (1) $\implies$ (2) implication of Claim D.3 implies that $V^f(p - s) \geq V^f(p)$. Letting $q = p' - s$, we have that $p - s = p \wedge q$ and that $p' = p \vee q$. As $p_{\Omega_{\to f}} \leq p'_{\Omega_{\to f}} = q_{\Omega_{\to f}}$, WQ implies that $V^f(p' - s) \geq V^f(p')$. The (3) $\implies$ (1) implication of Claim D.3 therefore implies that there exists $\Xi' \in \mathbf{D}^f(p')$ with $\Gamma \cap \Xi' = \varnothing$. By construction, we have that $\omega \notin \Xi'$ for all $\omega \in \Omega_f$ with $p'_\omega = -\infty$. Therefore, we must have that

$$\Xi' \cap \{\omega \in \Omega_{f\to} \mid p_\omega = p'_\omega\} \subseteq \Xi,$$

as desired.

We next prove that WQ implies DFSS-D. To derive this implication, we exploit Claim D.4. Let $q \geq q' \in \mathsf{P}^f$ and let $\Xi \in \mathbf{D}^f(q)$. Suppose that $\omega \in \Xi_{f\to}$ is such that $q_\omega = q'_\omega$. By construction, we must have that $q_\omega \in \mathbb{R}$. By the (1) $\implies$ (2) implication of Claim D.4, we have that $V^f(q_{\Omega_f \smallsetminus \{\omega\}}, s_\omega) > V^f(q)$ for all $s > q_\omega$. Letting $p = (q'_{\Omega_f \smallsetminus \{\omega\}}, s_\omega)$ and noting $p_{\Omega_{\to f}} = q'_{\Omega_{\to f}} \leq q_{\Omega_{\to f}}$, that $(q_{\Omega_f \smallsetminus \{\omega\}}, s_\omega) = p \vee q$, and that $q' = p \wedge q$, WQ implies that $V^f(q'_{\Omega_f \smallsetminus \{\omega\}}, s_\omega) > V^f(q')$ for all $s > q_\omega = q'_\omega$. The (2) $\implies$ (1) implication of Claim D.4 therefore implies that there exists $\Xi' \in \mathbf{D}^f(q')$ with $\omega \in \Xi'$, as desired.

E Other cooperative solution concepts

In this appendix, we examine the relationships between competitive equilibrium, trail stability and other cooperative solution concepts for matching in trading networks—formally developing the results that we described in Section 6. We start by deriving general properties of stability and chain stability: we show that under FS, stable and chain-stable outcomes are trail-stable, and that under FS and BCV, they lift to competitive equilibria. We then consider vertical supply chain settings—showing that stable and trail-stable outcomes coincide under FS and obtaining sufficient conditions for the existence of stable outcomes. Finally, we analyze trading networks without frictions, providing conditions under which competitive equilibrium outcomes, stable outcomes, chain-stable outcomes, and strongly group stable outcomes coincide with one another and with trail-stable outcomes.

E.1 Stability and chain stability in trading networks

As we have already defined stability (in Section 5.1), we begin by defining chain stability. Chains are the sets that consist of all of the contracts in a trail. A *blocking chain* is a blocking set (in the sense of Definition 3) that is a chain. Chain stability rules out the existence of blocking chains.

Definition E.1 (Ostrovsky, 2008; Hatfield et al., 2018)**.** A *chain* is a set of contracts of the form $\{z_1, \ldots, z_n\}$, where $(z_1, \ldots, z_n)$ is a trail. An outcome is *chain-stable* if it is individually rational and there is no blocking chain.

Under FS, it turns out that stability and chain stability refine trail stability.

Proposition E.1. *Under FS, every chain-stable outcome is trail-stable.*

Proposition E.1 is a version of Proposition 8 in Fleiner et al. (2018) for settings with continuous prices. To prove Proposition E.1, we adapt Fleiner et al.'s (2018) argument to our setting. Formally, a *locally blocking circuit* is a circuit in which every pair of adjacent contracts is demanded by their common firm in every choice set.

Definition E.2. Let Y be an outcome. A sequence of contracts $(z_1, \ldots, z_n)$ is a *locally blocking circuit* if:

- for all $1 \leq i \leq n$, we have $\{z_{i-1}, z_i\} \subseteq W$ for all $W \in C^{f_i}\left(Y_{f_i} \cup \{z_{i-1}, z_i\}\right)$, where $f_i = \mathsf{s}(z_i) = \mathsf{b}(z_{i-1})$.

Here, we write $z_0 = z_n$.

To prove Proposition E.1, we show (as in Fleiner et al., 2018) that every shortest locally blocking circuit or locally blocking trail gives rise to a blocking chain; Proposition E.1 follows directly from this claim.

Claim E.1. *Let Y be an individually rational outcome. Under FS, if $(z_1, \ldots, z_n)$ is shortest among all locally blocking circuits and locally blocking trails for Y, then the chain $\{z_1, \ldots, z_n\}$ blocks Y.*

Proof. We prove the contrapositive of the claim. Suppose that $(z_1, \ldots, z_n)$ is a locally blocking circuit or locally blocking trail but that $Z = \{z_1, \ldots, z_n\}$ does not block Y. Then, there is a firm f, a contract $z_j \in Z_f$, and a set $W \in C^f(Y_f \cup Z_f)$ with $z_j \notin W$. Without loss of generality, we can assume that $f = \mathsf{s}(z_j)$, so $f = f_j$. We show that there is a locally blocking circuit or locally blocking trail that is shorter than $(z_1, \ldots, z_n)$.

By Theorem A.1, SFS must be satisfied. We divide into cases based on whether $j = 1$ and whether we have a trail or a circuit to complete the proof of the claim.

Case 1: $j = 1$ and $(z_1, \ldots, z_n)$ is a locally blocking trail. By IFSS, there exists $W' \in C^f(Y_f \cup Z_{f\to})$ with $z_1 \notin W'$. Among all such W', we take W to minimize $|W' \smallsetminus Y_f|$. As $Y_f \notin C^f(Y_f \cup \{z_1\})$, we have that $Y_f \notin C^f(Y_f \cup Z_{f\to})$, and hence that $W \not\subseteq Y_f$.

Let $z_k \in W \smallsetminus Y_f$ be arbitrary. The selection of W ensures that $W \smallsetminus Y_f \subseteq Y'$ for all $Y' \in C^f(W \cup Y_f)$, so in particular $z_k \in Y'$ for all $Y' \in C^f(W \cup Y_f)$. By IFSS, it follows that $z_k \in W_0$ for all $W_0 \in C^f(Y_f \cup \{z_k\})$, so $(z_k, \ldots, z_n)$ is a shorter locally blocking trail.

Case 2: $j \neq 1$ or $(z_1, \ldots, z_n)$ is a locally blocking circuit. In either case, the contract z_{j-1} is well-defined. By IFSS, there exists $W' \in C^f(Y_f \cup \{z_{j-1}\} \cup Z_{f\to})$ with $z_j \notin W'$. Among all such W', we take W to minimize $|W' \smallsetminus Y_f|$.

As $\{z_{j-1}, z_j\} \subseteq B$ for all $B \in C^f(Y_f \cup \{z_{j-1}, z_j\})$, we have that $z_{j-1} \in W'$ for all $W' \in C^f(Y_f \cup \{z_{j-1}\} \cup Z_{f\to})$ by DFSP. In particular, we have that $z_{j-1} \in W$.

Let $z_k \in W \smallsetminus Y_f$ be arbitrary. The selection of W ensures that $W \smallsetminus Y_f \subseteq Y'$ for all $Y' \in C^f(W \cup Y_f)$, so in particular $z_k \in Y'$ for all $Y' \in C^f(W \cup Y_f)$. By IFSS, it follows that $z_k \in B$ for all $B \in C^f(Y_f \cup \{z_{j-1}, z_k\})$. If $k < j$, then

$(z_k, \dots, z_{j-1})$ is a shorter locally blocking circuit. If $k > j$ and $(z_1, \dots, z_n)$ is a locally blocking circuit (resp. locally blocking trail), then $(z_1, \dots, z_{j-1}, z_k, \dots, z_n)$ is a shorter locally blocking circuit (resp. locally blocking trail).

The cases exhaust all possibilities, completing the proof of the claim. □

The next example shows that FS is generally needed for stable or chain-stable outcomes to be trail-stable—even if there are no technological constraints or distortionary frictions.

Example E.1 (Stable outcomes may not be trail-stable without FS). As depicted in Figure 1(a), there are two firms, f_1 and f_2, which interact via two trades, ζ and ψ. There are no taxes. The firms have quasilinear utility functions (see (1)) with valuation functions defined by

$$\begin{aligned} v^{f_1}(\varnothing) = v^{f_2}(\varnothing) &= 0 \\ v^{f_1}(\{\zeta\}) = v^{f_1}(\{\psi\}) &= 1 \\ v^{f_1}(\{\zeta, \psi\}) &= -100 \\ v^{f_2}(\{\zeta\}) = v^{f_2}(\{\psi\}) &= -100 \\ v^{f_2}(\{\zeta, \psi\}) &= 1. \end{aligned}$$

The autarky outcome is stable, as no non-empty set of contracts is individually rational for both f_1 and f_2. However, the trail $((\zeta, 0), (\psi, 0))$ locally blocks the autarky outcome. Thus, the autarky outcome is stable but not trail-stable.

Note that trades ζ and ψ are not (cross-side) complementary for firm f_1, which implies that f_1's preferences are not fully substitutable. □

We next show that stable and chain-stable outcomes lift to competitive equilibria under the conditions for the existence of competitive equilibria.

Theorem E.1. *Under FS and BCV, chain-stable outcomes lift to competitive equilibria.*

It follows that stable outcomes lift to competitive equilibria under FS and BCV—generalizing Theorem 6 in Hatfield et al. (2013) to trading networks with distortionary frictions and income effects. Note however that stable and chain-stable outcomes do not generally exist in our model (even under FS and BCV).

To prove Theorem E.1, we consider a chain-stable outcome A. We construct a modified economy (as in the proof of Theorem 1)—in which FS and BWP are satisfied—by giving every firm the option to make any trade by paying a large cost Π. Unlike in the proof of Theorem 1, we choose Π to depend the prices in A. We show that A is chain-stable in the modified economy. Proposition E.1 implies that A is trail-stable in the modified economy, and hence Theorem 3 guarantees that A lifts to a competitive equilibrium in the modified economy. To complete the proof, we use the result (that we proved in Appendix B) that every competitive equilibrium in the modified economy is in fact a competitive equilibrium in the original economy under BCV for Π sufficiently large. We cannot apply the arguments from Hatfield et al. (2013) to prove Theorem E.1 because those arguments rely on the efficiency of competitive equilibria in a modified economy and frictions generally make competitive equilibria inefficient (see Footnote 24 in the paper).

Formally, let A be any chain-stable outcome, and let $\Xi = \tau(A)$. For $\omega \in \tau(A)$, let p_ω be the unique price such that $(\omega, p_\omega) \in A$.

As in Appendix B, let

$$\mathcal{K}^f = - \inf_{u^f(\Xi,t) \geq u^f(\varnothing,0)} \sum_{\omega \in \Omega_f} t_\omega$$

for $f \in F$, which is finite by BCV. Define a quantity

$$\Pi = 1 + \sum_{f \in F} \mathcal{K}^f + 2 \sum_{\omega \in \Xi} |p_\omega|.$$

Recall the definition of $\widehat{u}^f : \mathcal{P}(\Omega_f) \times \mathbb{R}^{\Omega_f} \to \mathbb{R}$ from Appendix B, which is

$$\widehat{u}^f(\Xi, t) = \max_{\Xi \subseteq \Psi \subseteq \Omega_f} u^f\left(\Psi, \left(t_{\Omega_f \smallsetminus \Psi \cup \Xi}, (t - \Pi)_{\Psi \smallsetminus \Xi}\right)\right).$$

Consider a modified economy in which utility functions are given by $\widehat{u}^f$ for $f \in F$. The following claim asserts that A is a chain-stable outcome in the modified economy.

Claim E.2. *Under BCV, if A is chain-stable in the original economy, then A is chain-stable in the modified economy.*

Proof. The outcome A is clearly individually rational in the modified economy. It remains to prove that A is not blocked by any chain in the modified economy. Suppose for the sake of deriving a contradiction that there is a blocking chain Z in the modified economy.

Let $\widehat{C}^f$ and $\widehat{U}^f$ denote f's choice correspondence and utility function over sets of contracts, respectively, in the modified economy. For $f \in F$ and $Y^f \in \widehat{C}^f(A_f \cup Z_f)$, note that $\widehat{U}^f\left(Y^f\right) \geq \widehat{U}^f(\varnothing)$. Hence, we have that

$$-\mathcal{K}^f \leq \sum_{(\omega,p'_\omega)\in Y^f_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega)\in Y^f_{\to f}} p'_\omega \leq \sum_{(\omega,p'_\omega)\in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega)\in Z_{\to f}} p'_\omega + \sum_{\omega\in\Xi_f} |p_\omega|,$$

where the first inequality is due to Lemma B.3(a). It follows that

$$\sum_{(\omega,p'_\omega)\in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega)\in Z_{\to f}} p'_\omega + \sum_{\omega\in\Xi_f} |p_\omega| + \mathcal{K}^f \geq 0.$$

But note that

$$\sum_{f\in F}\left(\sum_{(\omega,p'_\omega)\in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega)\in Z_{\to f}} p'_\omega + \sum_{\omega\in\Xi_f} |p_\omega| + \mathcal{K}^f\right) = 2\sum_{\omega\in\Xi} |p_\omega| + \sum_{f\in F}\mathcal{K}^f = \Pi - 1.$$

It follows that

$$\sum_{(\omega,p'_\omega)\in Z_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega)\in Z_{\to f}} p'_\omega + \sum_{\omega\in\Xi_f} |p_\omega| + \mathcal{K}^f \leq \Pi - 1 < \Pi$$

for all $f \in F$, so

$$\sum_{(\omega,p'_\omega)\in Y^f_{f\to}} p'_\omega - \sum_{(\omega,p'_\omega)\in Y^f_{\to f}} p'_\omega \leq -\mathcal{K}^f + \Pi - 1 < -\mathcal{K}^f + \Pi.$$

Hence, the contrapositive of Lemma B.3(b) implies that $\widehat{U}^f\left(Y^f\right) \leq U^f\left(Y^f\right)$ for all $f \in F$. By the definition of $\widehat{u}^f$, we must therefore have that $\widehat{U}^f\left(Y^f\right) = U^f\left(Y^f\right)$.

Let $W \in C^f(A_f \cup Z_f)$ be arbitrary. In light of the previous paragraph and the fact that $U^f(W) \leq \widehat{U}^f(W)$, we must have that $W \in \widehat{C}^f(A_f \cup Z_f)$. Since Z blocks A in the modified economy, we must have that $Z_f \subseteq W$. Hence, Z blocks A in the original economy—contradicting the hypothesis that A is chain-stable in the original economy. $\square$

Proof of Theorem E.1. Claim E.2 guarantees that A is chain-stable in the modified economy. By Proposition E.1, A is trail-stable in the modified economy. Lemmata B.1 and B.2 ensure that FS and BCV are satisfied in the modified economy. Hence, by Theorem 3, there exists a competitive equilibrium $[\Xi; p]$ in the modified economy with $\kappa([\Xi; p]) = A$. Lemma B.4 guarantees that $[\Xi; p]$ is a competitive equilibrium in the modified economy. □

Hatfield et al. (2013) show that stable outcomes need not lift to competitive equilibria without FS (see Example 1 in Hatfield et al. (2013)). It turns out that FS is not sufficient for stable outcomes to lift to competitive equilibria—essentially for the same reason that FS is not sufficient for the existence of competitive equilibria.

Example E.2 (Stable outcomes need not lift to competitive equilibria under FS alone). Consider the trading network from Example 3. The autarky outcome is stable, because s is not willing to trade at any price. As there are no competitive equilibria, there is a stable outcome that does not lift to a competitive equilibrium. □

E.2 Supply chains

In supply chains, or acyclic trading networks, no firm can be simultaneously upstream and downstream from another firm even via intermediaries (Ostrovsky, 2008; Westkamp, 2010; Hatfield and Kominers, 2012).

Assumption E.1 (Acyclicity—AC)**.** There do not exist $n \geq 1$ and trades $\omega_1, \ldots, \omega_n$ such that $\mathsf{s}(\omega_i) = \mathsf{b}(\omega_{i-1})$ for all $1 \leq i \leq n$, where $\omega_0 = \omega_n$.

As shown by Ostrovsky (2008) and Hatfield and Kominers (2012), imposing acyclicity can help ensure the existence of stable outcomes in trading networks with discrete and bounded prices. In supply chains, trail stability is equivalent to chain stability (see, e.g., Fleiner et al. (2018)). The following lemma relates stability and chain/trail stability in supply chains.

Lemma E.1. *Under FS and AC, every trail-stable outcome is stable.*

Proof. The proof is similar to the proof of Theorem 7 in Hatfield and Kominers (2012). By Theorem A.1 in Appendix A, we can assume that SFS is satisfied.

We prove the contrapositive. Let A be outcome that is not stable. If A is not individually rational, then clearly A is not trail-stable. Thus, we can assume that A is blocked by a non-empty blocking set Z.

Since Z is non-empty and the trading network is acyclic, there is a firm f_1 with $Z_{\to f_1} = \varnothing$ and $Z_{f_1 \to} \neq \varnothing$. Let $z_1 \in Z_{f_1 \to}$ be arbitrary. By IFSS, we have that $z_1 \in Y$ for all $Y \in C^{f_1}(A_{f_1} \cup \{z_1\})$. Let $f_2 = \mathsf{b}(z_1)$.

If $z_1 \in Y$ for all $Y \in C^{f_2}(A_{f_2} \cup \{z_1\})$, then (z_1) is a locally blocking trail. Hence, we can assume that $z_1 \notin Y$ for some $Y \in C^{f_2}\left(A_{f_2} \cup \{z_1\}\right)$. By revealed preference, we must have that $A_{f_2} \in C^{f_2}(A_{f_2} \cup \{z_1\})$. DFSP implies that $z_1 \in W'$ for all $W' \in C^{f_2}(A_{f_2} \cup \{z_1\} \cup Z_{f_2 \to})$. Let $W \in C^{f_2}(A_{f_2} \cup \{z_1\} \cup Z_{f_2 \to})$ minimize $|W' \smallsetminus A|$ among all $W' \in C^{f_2}(A_{f_2} \cup \{z_1\} \cup Z_{f_2 \to})$. By IFSS, we must have that $W = \{z_1, z_2\}$ for some $z_2 \in Z_{f_2 \to}$. Note that $\{z_1, z_2\} \subseteq Y$ for all $Y \in C^{f_2}(A_{f_2} \cup \{z_1, z_2\})$ by construction.

A similar argument to the previous paragraph shows that (z_1, z_2) is a locally blocking trail or there exists $z_3 \in Z$ with $\mathsf{s}(z_3) = \mathsf{b}(z_2)$ such that $\{z_2, z_3\} \subseteq Y$ for all $Y \in C^{f_2}(A_{f_2} \cup \{z_2, z_3\})$. By induction and due to acyclicity, we obtain a locally blocking trail. Hence, A is not trail-stable. □

Proposition E.1 and Lemma E.1 imply that trail-stable, stable, and chain-stable outcomes coincide in supply chains under FS, yielding a continuous-price version of Theorem 7 in Hatfield and Kominers (2012).

Corollary E.1. *Under FS and AC, trail-stable outcomes, stable outcomes, and chain-stable outcomes coincide.*

Proof. Trail-stable outcomes are stable by Lemma E.1. Stable outcomes are always chain-stable. Chain-stable outcomes are trail-stable by Proposition E.1. □

Combining Corollary E.1 with our results on trading networks with frictions, we obtain that competitive equilibrium, trail stability, stability, and chain stability are all essentially equivalent in supply chains (under FS and BCV).

Corollary E.2. *Under FS, BCV, and AC, competitive equilibrium outcomes, trail-stable outcomes, stable outcomes, and chain-stable outcomes exist and coincide.*

Proof. Competitive equilibria exist by Theorem 1. Competitive equilibrium outcomes are trail-stable by Theorem 2. Trail-stable outcomes, stable outcomes, and chain-stable outcomes coincide by Corollary E.1. Chain-stable outcomes lift to competitive equilibria by Theorem E.1. □

Corollary E.2 implies that stable and chain-stable outcomes exist in supply chains under FS and BCV. This existence result is a version of Theorem 1 in Ostrovsky (2008) and Theorem 3 in Hatfield and Kominers (2012) for settings in which prices are continuous. However, Corollary E.2 holds even when willingness to pay is unbounded (i.e., BWP is not satisfied), unlike the existence results that Ostrovsky (2008) and Hatfield and Kominers (2012) prove.

E.3 Trading networks without frictions

In the presence of distortionary frictions, firms have different marginal rates of substitution between forms of transfer. For example, in the presence of transaction taxes, all firms find reductions in outgoing payments more desirable than equal increases in incoming payments.

We formalize "equalization of marginal rates of substitution between forms of transfer" as "indifference between all forms of transfer" in defining our "no frictions" condition. Intuitively, if the firms share the same marginal rates of substitution between forms of transfer, then transfers can be redenominated so that the marginal rates of substitution become 1. The possibility of redenomination is precisely why, for example, the presence of multiple currencies does not cause frictions *per se*.

Assumption E.2 (No frictions—NF)**.** For all $f \in F$ and $t, t' \in \mathbb{R}^{\Omega_f}$ with $\sum_{\omega \in \Omega_f} t_\omega = \sum_{\omega \in \Omega_f} t'_\omega$, we have that $u^f(\Xi, t) = u^f(\Xi, t')$ for all $\Xi \subseteq \Omega_f$.

Recall that, in Examples 1 and 2, paying one unit is more costly for firms than receiving one unit (due to transaction taxes). NF rules out these differences in the costs of transfers and requires that firms only care about the net transfers that they receive or pay. Therefore, NF requires that a unit of transfer for one trade be equivalent to a unit of transfer for any other trade. In particular, any transferable utility economy satisfies NF. Under NF, we can write $u^f(\Xi, t) = \tilde{u}^f(\Xi, \tilde{t})$, where $\tilde{t} = \sum_{\omega \in \Omega} t_\omega$ is the net transfer.

While NF rules out distortionary frictions—such as variable transaction taxes and commissions—utility does not need to be perfectly transferable under NF because income effects are still permitted. For example, terminal buyers and sellers (i.e., firms who only buy or only sell) with unit demand can experience arbitrary income effects under FS and NF because full substitutability holds automatically for terminal firms that make only trade. Income effects are also possible for terminal firms with multi-unit demand.

Remark E.1. However, under FS and NF, firms cannot experience income effects along the locus of prices at which they have multi-unit demand for both upstream and downstream trades. To see why, note that NF requires that the income effect associated to a small decrease in the price of a demanded upstream trade ζ equal the income effect associated to a small increase in the price of a demanded downstream trade ψ. However, FS (or, equivalently, FS-D from Online Appendix D.1) requires that the income effect associated to a decrease in the price of ζ reduce demand for other upstream trades and raise demand for downstream trades. Conversely, FS requires that the income effect associated to an increase in the price of ψ raise demand for upstream trades and reduce demand for other downstream trades. When firms have multi-unit demand for both upstream and downstream trades, there are in fact upstream (resp. downstream) trades other than ζ (resp. ψ) whose demand can be reduced. Hence, as the two income effects must coincide under NF, the FS and NF conditions combine to rule out income effects along the locus of prices at which they have multi-unit demand for both upstream and downstream trades. □

We begin our analysis of trading networks without frictions by recalling the definition of strong group stability, which is the most demanding cooperative solution concept from the literature on matching with contracts. A strongly group stable outcome is immune to blocks by coalitions of firms that can commit to better, new contracts and maintain any existing contracts with each other and with firms outside the blocking coalition.

Definition E.3 (Hatfield et al., 2013)**.** An outcome A is *strongly unblocked* if there do not exist a non-empty set $Z \subseteq X \smallsetminus A$ and sets of contracts $Y^f \subseteq A_f \cup Z_f$ for $f \in F$ such that $Y^f \supseteq Z_f$ and $U^f\left(Y^f\right) > U^f\left(A_f\right)$ for all $f \in F$ with $Z_f \neq \varnothing$. An outcome is *strongly group stable* if it is individually rational and strongly unblocked.

In Definition E.3, Y^f is the set of contracts that f signs in the block. Note that Y^f need not be f's best choice from the set of available contracts. In particular, strong group stability rules out blocks in which firms only improve their utility by selecting all of the blocking contracts. Hence, as Hatfield et al. (2013) show, strong group stability is stronger than stability. Moreover, Y^f can contain existing contracts that the counterparties no longer want. In particular, strong group stability rules out blocks in which different members of the blocking coalition can make selections from the set of existing contracts that are incompatible with one another or involve firms outside the coalition. Hence, strong group stability also refines properties such as (strong) setwise stability (Echenique and Oviedo, 2006; Klaus and Walzl, 2009) and the core. As pointed out by Hatfield et al. (2013), strong group stability also refines strong stability (Hatfield and Kominers, 2015) and group stability (Konishi and Ünver, 2006).

Our next result shows that competitive equilibria are strongly group stable in trading networks without frictions.

Theorem E.2 (First Welfare Theorem)**.** *Under NF, competitive equilibrium outcomes are strongly group stable.*

Theorem E.2 extends Theorem 5 in Hatfield et al. (2013)—which shows that competitive equilibrium outcomes are strongly group stable—to settings with income effects. Since strongly group stable outcomes are stable and in the core, Theorem E.2 implies that competitive equilibrium outcomes are stable and in the core in trading networks without frictions. As core outcomes are Pareto-efficient, Theorem E.2 is a version of the First Welfare Theorem (Debreu, 1951).

Proof. We prove the contrapositive. Let $[\Xi; p]$ be an arrangement and suppose that $A = \kappa([\Xi; p])$ is not strongly group stable. If A is not individually rational, then clearly $[\Xi; p]$ is not a competitive equilibrium. Thus, we can assume that A is not strongly unblocked—that is, that there exists a non-empty set of contracts $Z \subseteq X \smallsetminus A$ and, for each $f \in F$ with $Z_f \neq \varnothing$, a set of contracts $Y^f \subseteq Z_f \cup A_f$ with $Y^f \supseteq Z_f$ and $U^f\left(Y^f\right) > U^f\left(A_f\right)$ (see Definition E.3).

We let $F' = \{f \in F \mid Z_f \neq \varnothing\}$. For each $f \in F'$, we let

$$\mathcal{M}^f = \sup\left\{\tilde{t} \;\middle|\; \tilde{u}^f\left(\tau(Y^f), \sum_{\omega\in\tau(Y^f)_{f\to}} p_\omega - \sum_{\omega\in\tau(Y^f)_{\to f}} p_\omega - \tilde{t}\right) \geq U^f\left(A_f\right)\right\}$$

denote the negative of the compensating variation for f from the change from $\tau(A_f)$ to $\tau(Y^f)$ at price vector p. For $\omega \in \tau(Z)$, let q_ω be the unique price such that

$(\omega, q_\omega) \in Z$. Define $q_\omega = p_\omega$ for $\omega \in \Omega \smallsetminus \tau(Z)$. The definition of Y^f ensures that

$$\tilde{u}^f \left(\tau(Y^f), \sum_{\omega \in \tau(Y^f)_{f \to}} q_\omega - \sum_{\omega \in \tau(Y^f)_{\to f}} q_\omega \right) > U^f\left(A_f\right)$$

for all $f \in F'$. It follows that

$$\begin{aligned} \mathcal{M}^f &> \sum_{\omega \in \tau(Y^f)_{f \to}} p_\omega - \sum_{\omega \in \tau(Y^f)_{\to f}} p_\omega - \sum_{\omega \in \tau(Y^f)_{f \to}} q_\omega + \sum_{\omega \in \tau(Y^f)_{\to f}} q_\omega \\ &= \sum_{\omega \in \tau(Y^f)_{f \to}} (p_\omega - q_\omega) + \sum_{\omega \in \tau(Y^f)_{\to f}} (q_\omega - p_\omega). \end{aligned}$$

Because $p_\omega = q_\omega$ for $\omega \notin Z$ and $Z_f \subseteq Y^f$, we have that

$$\mathcal{M}^f > \sum_{\omega \in \tau(Z_f)_{f \to}} (p_\omega - q_\omega) + \sum_{\omega \in \tau(Z_f)_{\to f}} (q_\omega - p_\omega).$$

Summing over $f \in F'$, we have that $\sum_{f \in F'} \mathcal{M}^f > 0$. Thus, there exists $f \in F'$ with $\mathcal{M}^f > 0$. For such f, we have that

$$\begin{aligned} u^f\left(\tau(Y^f), \left(p_{\tau(Y^f)_{f \to}}, (-p)_{\tau(Y^f)_{\to f}}, 0_{\Omega_f \smallsetminus \tau(Y^f)}\right)\right) &> U^f\left(A_f\right) \\ &= u^f\left(\Xi_f, \left(p_{\Xi_{f \to}}, (-p)_{\Xi_{\to f}}, 0_{\Omega_f \smallsetminus \Xi}\right)\right), \end{aligned}$$

so $\Xi_f \notin D^f\left(p_{\Omega_f}\right)$. Therefore, $[\Xi; p]$ is not a competitive equilibrium. □

Combining Theorem E.2 with our results on trading networks with frictions, we obtain that all of the solution concepts described in this paper are essentially equivalent in trading networks without frictions (under FS and BWP).

Corollary E.3. *Under FS, BWP, and NF, competitive equilibrium outcomes, trail-stable outcomes, stable outcomes, chain-stable outcomes, strongly group stable outcomes exist and coincide.*

Proof. Competitive equilibrium outcomes exist and coincide with trail-stable outcomes by Corollary 2, and are strongly group stable by Theorem E.2. Strongly group stable outcomes are always stable, and stable outcomes are always chain-stable. Chain-stable outcomes are trail-stable by Proposition E.1. □

Under NF, we can also restate BCV more simply using only net transfers, since firms are indifferent regarding the sources of transfers.

Assumption 2′ (Bounded CVs under NF—BCV-NF)**.** For all $f \in F$, we have

$$\inf_{(\Xi,\tilde{t})|\tilde{u}^f(\Xi,\tilde{t})\geq\tilde{u}^f(\varnothing,0)} \tilde{t} > -\infty.$$

In frictionless markets, under FS and BCV-NF, we obtain an equivalence between competitive equilibrium, stability, chain stability, and strong group stability.

Corollary E.4. *Under FS, BCV-NF, and NF, competitive equilibrium outcomes, stable outcomes, chain-stable oucomes, strongly group stable outcomes exist and coincide.*

Proof. Competitive equilibria exist by Theorem 1. Competitive equilibrium outcomes are strongly group stable by Theorem E.2. Strongly group stable outcomes are always stable, and stable outcomes are always chain-stable. Chain-stable outcomes lift to competitive equilibria by Theorem E.1. □

Corollary E.4 generalizes Theorem 5 and the first part of Theorem 9 in Hatfield et al. (2013), which deals with transferable utility economies. Ikebe et al. (2015), Candogan et al. (2017), and Hatfield et al. (2018) prove similar equivalence results for transferable utility economies. In contrast, Corollary E.4 applies in the presence of income effects.

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